Chapter 6.2 One-to-One Functions; Inverse Functions

One-to-One Functions

- A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range.
  
  \[
  \text{If } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)
  \]

- **Horizontal-line Test:**
  If every horizontal line intersects the graph of a function \( f \) in at most one point, then \( f \) is one-to-one

Determine whether the relation is a function. If the relation is a function, state whether the function is one-to-one

![Graphs of one-to-one functions and their corresponding horizontal-line tests.](image)

Inverse Functions

- Suppose that \( f \) and \( g \) are two functions such that
  
  \[
  f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g.
  \]
  
  and
  
  \[
  g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.
  \]
  
  The function \( g \) is the **inverse of the function \( f \)**, denoted by \( f^{-1} \).

- An **inverse function of \( f \)** is the function in which the correspondence from the range of \( f \) back to the domain of \( f \).

- **Domain of \( f = Range \text{ of } f^{-1}; \text{ Range of } f = Domain \text{ of } f^{-1}.**

- If a function \( f \) is one-to-one, then the function has an inverse function \( f^{-1} \).

- The graph of the inverse function \( f^{-1} \) is symmetric with respect to the line \( y = x \).

- To find the inverse of a function, interchange the variables \( x \) and \( y \) and then solve for \( y \).
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Exercises

1. Is the following set of ordered pairs a function?
   \[ \{(-4, -1), (7, 9), (-2, -2), (9, 8)\} \]
   Does the given set represent a function?
   - Yes
   - No
   (Solution 1)
   Because there is no ordered pair that has same first element, the set of ordered pairs is a function.
   If a first element in a relation is repeated, the relation is not a function.

2. What is the domain of \( f(x) = \frac{x + 5}{x^2 + 4x - 21} \)?
   Select the correct choice below and, if necessary, fill in the answer box(es) to complete your choice.
   - A. The domain is \( \{x | x \geq 3, x \neq 3\} \).
   - B. The domain is \( \{x | x \neq 7, 3\} \).
   - C. The domain is \( \{x \} \).
   - D. The domain is \( \{x | x \leq 7, x \neq 7\} \).
   - E. The domain is the set of all real numbers.
   (Solution 2)
   \[ x^2 + 4x - 21 = 0 \]
   Step 1: Factor out GCF. GCF = 1
   Step 2: Find \( a, b, c \)
   \[ a = 1, b = 4, c = -21 \]
   Step 3: Find \( ac \)
   \[ ac = 1 \cdot (-21) = -21 \]
   Step 4: Find positive factors of \( ac \) whose products is \( ac \) and whose sum is \( b \).
   Positive factors are \((1 \times 21), (3 \times 7)\).
   Since \( ac \) is negative, one is positive and the other is negative.
   Since \( b \) is positive, the bigger factor is positive.
   Thus, \((-1 \times 21), (-3 \times 7)\).
   Step 5: Rewrite the middle term, \( bx \), using two numbers obtained in the step 4 as coefficients. Then factor out the result by grouping.
   \[ x^2 + 4x - 21 = x^2 - 3x + 7x - 21 \]
   \[ = (x^2 - 3x) + 7(x - 3) \]
   \[ = x(x - 3) + 7(x - 3) \]
   \[ = (x - 3)(x + 7) \]
   Thus, \[ x^2 + 4x - 21 = 0 \]
   \[ (x - 3)(x + 7) = 0 \]
   \[ x = 3, -7 \]
   Thus, the domain is \( \{x | x \neq 3, -7\} \).

3. Complete the sentence below.
   If every horizontal line intersects the graph of a function at no more than one point, \( f \) is a(n) __function.

4. The graph of a function \( f \) is given. Use the horizontal-line test to determine whether \( f \) is one-to-one.
   Is \( f \) one-to-one? Choose the correct answer below.
   - Yes
   - No
   (Solution 4)
   The function \( f \) is one-to-one because every horizontal line intersects the graph of a function \( f \) at most one point.
5. Find the inverse of the one-to-one function.
State the domain and range of the inverse function.
\{(-7, 0), (1, 12), (2, -3), (3, 5), (4, 1)\}

Which of the following is the inverse function?
- A: \{(-7, 0), (1, 12), (2, -3), (3, 5), (4, 1)\}
- B: \{(-7, 0), (1, 12), (2, -3), (3, 5), (4, 1)\}
- C: \{-7, 0\}
- D: \{(-7, 0), (1, 12), (2, -3), (3, 5), (4, 1)\}

What is the domain of the inverse function?
- A: \{0, 1, 2, 3, 4\}
- B: \{-7, 12, -3, -5, 11\}
- C: \{0, 1, 2, 3, 4\}
- D: \{-7, 12, -3, -5, 11\}

What is the range of the inverse function?
- A: \{0, 1, 2, 3, 4\}
- B: \{-7, 12, -3, -5, 11\}
- C: \{0, 1, 2, 3, 4\}
- D: \{-7, 12, -3, -5, 11\}

6. The function \(f(x) = \frac{8 - 4x}{x}\) is one-to-one.
Find its inverse and check your answer.

\[
f^{-1}(x) = \frac{8 - 4x}{x}
\]

(Solution 6)
In \(y = f(x)\), interchange the variables \(x\) and \(y\) and then solve for \(y\) to find the inverse of a function.

\[
f(x) = \frac{8 - 4x}{x} \Rightarrow y = \frac{8}{4 + x} \Rightarrow x = \frac{8}{4 + y} \\
x(4 + y) = 8 \Rightarrow 4x + xy = 8 \\
-4x = -4x \\
xy = 8 - 4x \\
x = \frac{xy}{x} = \frac{8 - 4x}{x} \\
y = \frac{8 - 4x}{x}
\]

7. The function \(f(x) = \frac{5x}{x + 6}\) is one-to-one.
Find its inverse and check your answer.

\[
f^{-1}(x) = \frac{6x}{5 - x}
\]

(Solution 7)
In \(y = f(x)\), interchange the variables \(x\) and \(y\) and then solve for \(y\) to find the inverse of a function.

\[
f(x) = \frac{5x}{x + 6} \Rightarrow y = \frac{5x}{x + 6} \Rightarrow x = \frac{5y}{y + 6} \\
x(y + 6) = 5y \Rightarrow xy + 6x = 5y \\
yx = 6x = 5y \\
x = \frac{6x}{5 - x} \\
y = \frac{6x}{5 - x}
\]

8. The function \(f(x) = \frac{4x}{3x - 7}\) is one-to-one.
Find its inverse and check your answer.

\[
f^{-1}(x) = \frac{-7x}{4 - 3x}
\]

(Solution 8)
In \(y = f(x)\), interchange the variables \(x\) and \(y\) and then solve for \(x\) to find the inverse of a function.

\[
y = \frac{4x}{3x - 7} \Rightarrow x = \frac{4y}{3y - 7} \\
x(3y - 7) = 4y \Rightarrow 3xy - 7x = 4y \\
-3xy = -3xy \\
-7x = 4y - 3xy \\
-7x = y(4 - 3x) \\
-7x = 4 - 3x \\
\frac{-7x}{4 - 3x} = y
\]
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9. The function \( f(x) = \frac{2x + 4}{3x - 7} \) is one-to-one. Find its inverse and check your answer.

\[
f^{-1}(x) = \frac{-7x - 4}{2 - 3x}
\]

(Solution 9)

\[
f(x) = \frac{2x + 4}{3x - 7} \Rightarrow y = \frac{2x + 4}{3} \Rightarrow x = \frac{3y - 7}{2y + 4}
\]

\[
x(3y - 7) = 2y + 4
\]

\[
3xy - 7x = 2y + 4
\]

\[
x - 2y
\]

\[
\frac{3xy - 2y - 7x}{3} = 4 + 7
\]

\[
\frac{y(3x - 2)}{3} = \frac{4 + 7x}{3x - 2}
\]

\[
y = \frac{3x - 2}{3x - 2}
\]

10. The function \( f(x) = \frac{2x + 8}{x + 8} \) is one-to-one. Find its inverse and check your answer.

\[
f^{-1}(x) = \frac{8x - 8}{2 - x}
\]

(Solution 10)

\[
f(x) = \frac{2x + 8}{x + 8} \Rightarrow y = \frac{2x + 8}{x + 8} \Rightarrow x = \frac{2y + 8}{y + 8}
\]

\[
x = \frac{2y + 8}{y + 8}
\]

\[
xy + 8x = 2y + 8
\]

\[
x = \frac{y + 8}{2y + 8}
\]

\[
x - 2y
\]

\[
x - 2y
\]

\[
\frac{y(x - 2)}{x - 2} = 8 - 8x
\]

\[
\frac{y(x - 2)}{x - 2} = \frac{8 - 8x}{x - 2}
\]

\[
y = \frac{8 - 8x}{x - 2}
\]